# Layers – An Abstract Extension of Fields for Higher Dimensions

H.M. Nasir* and M.S.M. Waasim

*Department of Mathematics, Faculty of Science, University of Peradeniya*

## Introduction

We propose a new abstract algebraic system, which extends the notion of real and complex number systems. With the relaxation of some field axioms, we are able to extend the complex number system to higher dimensions. This extension gives ways to analyze many problems in physics and engineering.

The study was motivated by considering the general solution of the Laplace equation. In a three dimensional domain, the Laplace equation is given by $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$ Separation of variables $u = X(x)Y(y)Z(z)$ yields $\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$ Since the terms in this equation are dependent on only one variable, each term equals to a constant, say, $\frac{X''}{X} = -a^2$, $\frac{Y''}{Y} = -b^2$ and $\frac{Z''}{Z} = -c^2$, where $a, b, c$ satisfy $a^2 + b^2 + c^2 = 0$, and the solutions are given by $u = e^{ax+by+cz}$. In fact, one may easily see that for any function $f(w)$ with $f''(w) \neq 0$, the expression $u = f(ax + by + cz)$ is a solution. We seek a space in which the constants might be with the conditions that $a, b, c$ are linearly independent and quadratic dependent. In fact, there are no constants in the real and/or complex fields that satisfy the above two properties, because any three member in these fields are linearly dependent. This observation demands us to consider a number system, which is other than the real and complex fields. This motivation and application was presented in IWAMCS07 (Nasir *et al.*, 2007).

## Literature review

The theory of numbers has always been a quite fascinating and rich subject and seems to stem from ancient Greece. Complex number theory is by itself a nice achievement in this endeavor. Extensions of complex numbers have been an active subject in the 19th century by many prominent mathematicians such as Hurwitz, Clifford, Graves, Grassman and Hamilton. The notable extensions in this period are the Clifford and Grassman algebras that have applications in physics. Hamilton's quaternion and the subsequent discovery of octonion were considered big achievements in the search of higher dimensional fields.

But all these extensions lacked one or more important properties of multiplication, namely commutative and associative.

Recently, Fleury *et al.*, (1993) have extended the complex number theory with these two properties intact to higher dimensional vector spaces and called them multicomplex numbers. These multicomplex numbers mostly behave like complex numbers and most of the algebraic and analytic properties were shown by Fleury *et al.*, (1995). Our definition is more general and includes the multicomplex number system as a simple example.

### *A New Abstract Set*

We define a new abstract space as follows:

*Definition 1:* Let $V_0, V_1, V_2, \ldots, V_n, \ldots$ be vector spaces over the real field $R$ with $\dim V_n = d_n, V_0 = R$ and $V = \bigcup_{n=0}^{\infty} V_n$. Then, $u, v \in V \Rightarrow u \in V_n,\ v \in V_m$ for some $m, n$. Define multiplication of vectors in $V$ satisfying the following four axioms:

L1: $uv \in V_{m+n} \subseteq V$.

L2: $uv = vu, \quad \forall u, v \in V$ .

L3: $(uv)w = u(vw), \quad \forall u, v, w \in V$.

L4: $1u = u, \quad \forall u \in V$, where $1 \in V_0$.

In this new construction, we relax the existence of multiplicative inverse. Addition between elements of different vector spaces is used for symbolic purpose only. We call the abstract space 'Layer'; each vector space $V_n \subseteq V$ a layer of mode $n$; the space $V_0$ the 'unit layer' which is $R$; the space $V_1$ the 'base layer' and the dimension of $V_1$ the 'base dimension'. All the zero vectors $0_m \in V_m$ are commonly called zero and are denoted by 0.

*Example 1:* Let $D_2$ be the set defined by $D_2 = \{ax+by; a^2+b^2=0, x, y \in R\}$ and $V_n = \{a^{n-1}z \mid z \in D_2\}$ for $n>0$ with $V_0 = R$. Let $V = \bigcup_{n=0}^{\infty} V_n$. Then, $V$ forms a Layer with the usual multiplication. Here, $\dim V_m = 2$ for each $m>0$ and the Layer becomes a field isomorphic to the complex field.

*Example 2:* Let $V_n = \{p_n(x); \deg p_n \le n\}$. Define multiplication on $V = \bigcup_{n=0}^{\infty} V_n$ $pq = p_n(x)q_m(x)$, $p_n \in V_n, p_m \in V_m$. Here, we have $V_n \subset V_m$ for $n<m$ and $\deg V_n = n+1$.

*Example 3:* Let $p_n$ be polynomials of degree less than or equal to $n$ and define $W_n = \{P=(p_n, p_{n-1}); p_n \in V_n, p_{n-1} \in V_{n-1}\}$, where $V_n$ are vector spaces of polynomials defined in Example 2 and let $W = \bigcup_{n=0}^{\infty} W_n$. Define a multiplication

$$PQ = (p_n, p_{n-1}).(q_m, q_{m-1})$$
$$= (p_n q_m - (1+x^2)p_{n-1}q_{m-1}, p_n q_{m-1} + p_{n-1}q_m).$$

Here, $\dim W_m = 2m+1$.

*Main Results*

Here we list the main theorems for this paper.

*Theorem 1:* Let $V$ be a Layer with the base layer $V_1$ and $B_1 = \{a_1, a_2, a_3, \cdots a_n\}$ be a basis for $V_1$. Then, for each layer $V_m$, a basis can be expressed in terms of the powers of total degree $m$ of basis vectors from $B_1$.

*Definition:* A Layer is called a Harmonic Layer if there is a basis $B_1 = \{a_1, a_2, a_3, \cdots a_n\}$ for the base layer $V_1$ such that $\sum_{i=1}^{n} a_i^2 = 0$.

*Theorem 2:* Let $V$ be a harmonic Layer with base $B_1 = \{a, b, c\}$. Then, the set of $2m+1$ monomials of degree $m$ given by

$$\{a^m, a^{m-1}b, a^{m-2}b^2, \cdots, b^m;$$
$$a^{m-1}c, a^{m-2}bc, \cdots, b^{m-1}c\}$$

is a maximal linearly independent set in $V_m$ and hence $\dim V_m = 2m+1$.

*An application*

One of the interesting applications of Layer concept is the representation of functions in more than two variables by harmonic functions (Nasir, 2007). Harmonics are solutions of the Laplace equation.

Try $u = (ax+by+cz)^l$ for a solution, where $a$, $b$, $c$ are independent constants. Then $a$, $b$, $c$ satisfy $a^2+b^2+c^2=0$. Hence, we see that the solutions of the Laplace equations can be expressed as powers of the vectors from the base layer of some Layer $V$. Expanding the power representation using the constraint and by Theorem 2, we see that each term in the expansion is a solution of the Laplace equation. For example, with $l=2$, we have,

$$u_2 = (ax+by+cz)^2$$
$$= a^2x^2 + b^2y^2 + c^2z^2$$
$$+ 2abxy + 2bcyz + 2acxz$$
$$= a^2(x^2-z^2) + b^2(y^2-z^2)$$
$$+ ab(2xy) + bc(2yz) + ac(2xz).$$

Since, by Theorem 2, the monomial set $\{a^2, ab, b^2; ab, bc, ac\}$ is a basis for the layer $V_2$. Applying Laplacian on $u_2$, we deduce that the functions $x^2-z^2, y^2-z^2$, $2xy$, $2yz$, $2xz$ are all solutions of the Laplacian equation. This is true for arbitrary power $l$ as well.

**Conclusions**

We have constructed an abstract extension of complex numbers which has a multiplication with four properties. We called the abstract set 'Layer' and showed that the real and complex fields are trivial and simple cases of a Layer called harmonic Layer. We presented an application of our abstract setting in the construction of harmonic functions. The definition is restricted to be primitive to meet our immediate needs, but could be refined with further properties. Further research is needed to find other properties to be satisfied by the Layers.

**Acknowledgements**

This research was partially supported by the Peradeniya University research grant RG/2006/48/S.

**References**

Fleury, N., Detrauberg, M.R. and Yamaleev, R.M. (1993) Commutative extended complex numbers and connected trigonometry, *Journal of Mathematical Analysis and Applications,* 181(2), 431-457.

Fleury, N., Detrauberg, M.R. and Yamaleev R.M. (1995) Extended complex number analysis and conformal-like transformations, *Journal of Mathematical Analysis and Applications,* 191(1), 118-136.

Nasir, H.M. (2007) Spherical harmonics transform in a non-polar coordinate system and application to Fourier series in 2-Sphere, *Mathematical Methods in the Applied Sciences, John-Wiley,* 30(14), 1843-1854.

Nasir, H.M., Faham, M.A.A.M. and Waasim, M.S.M. (2007) Power representation of spherical harmonics and fast discrete spherical transform on cubed sphere, *International Workshop on Applied Mathematics and Computational Science (IWAMCS07) Sept. 25-29, 2007*, The University of Electro-Communications, Tokyo, Japan.