Some Trends in Mathematical Research[1]

1. Introduction

MY first duty in this Inaugural Address is to pay a tribute to my predecessor in the Chair of Mathematics. Professor Gulaskharam retired after something like thirty-five or forty years' service as a teacher of mathematics at school and university, and I should like to take this opportunity to express the hope that he may enjoy a happy and peaceful retirement.

My second duty is to indicate briefly what I am going to say, and the reason for my rashness in choosing this particular topic, with its forbidding title. Now and then I come across some friend who asks me ' What is all this mathematical research ?' Has not mathematics been developed for so long that there can hardly be anything more to add to it ?'. This is an interesting question, and I think there are many here who would like to discuss it, to have a rough idea of the way in which the subject has been growing and expanding in recent years, to know what the mathematicians are busying themselves with nowadays.

Mathematics, while it is almost the oldest subject of study in the civilized world, is also the youngest, for new and flourishing branches are ever being born. It is a big problem to keep pace with these developments, there being so many specialised branches that any one person can know only a small fraction of the total knowledge that comes under the term ' mathematics '. If one looks at an examination paper of the Cambridge Mathematical Tripos—Part III, where real specialisation begins, one will notice that in each paper there will be something like twenty-five to thirty questions, and generally speaking a candidate will not understand more than four or five questions, and will be able to answer only one or two. I mention this just to indicate that the tree of mathematics has many branches. It will be appreciated that on an occasion like this I must restrict myself to a few selected topics and trends of their development.

2. Utility and Beauty

There is a general impression that the motive behind what the mathematicians do may be found in the many and varied practical applications to which mathematical knowledge may be directed. Lancelot Hogben, a biologist and a well known populariser of mathematics, says in one of his books

> ' Without a knowledge of mathematics, the grammar of size and order, we cannot plan the rational society in which there will be leisure for all and poverty for none '.

1. An inaugural address, delivered on 1st February, 1951 at King George's Hall, University of Ceylon, Colombo.

Admitting that mathematics if properly applied could make great contributions to a planned rational society, it must however be noted that by far the greater portion of what mathematicians would consider as real mathematics has hardly any direct utility of the type that Hogben has in mind. For example, if we consider the notion of number, the knowledge of a school boy who is well conversant with arithmetic may suffice for most practical applications. But a substantial amount of the most elegant and beautiful notions of mathematics have arisen in trying to give a precise logical definition of number, and this knowledge has hardly any direct utilitarian value. The merit of a mathematical piece of work should not be judged by its immediately apparent practical applications. Perhaps it was as a safeguard against the possible overemphasis of practical utility, and the cramping of mathematics that would result therefrom, that Cambridge mathematicians are supposed to have as their toast ' Here is to mathematics, and may it be of no damned use to anybody '. (Excuse bad language, Mr. Chairman, I am quoting from my elders). The late Professor Hardy, the acknowledged leader of British mathematics during the last fifty years, used to delight in the fact that none of his mathematical contributions has ever been or ever likely to be of the slightest practical use—use or misuse, he would add, remembering the misuse of mathematics as of other sciences in the arts of war. His special pleasure was that his work had elegance and beauty, and a permanence that is characteristic of real mathematics, the quality that saved Babylonian mathematics while Babylonian civilization perished, or Greek mathematics while Greek civilization perished. A great mathematician of the last century, Karl Weirstrass, the father of modern analysis, is known to have said :

> ' A mathematician who is also not something of a poet will never be a complete mathematician '.

Approached in this way, mathematics, and the other sciences too, would appear as genuine branches of *Litterae Humaniores*.

What I have been saying would justify my conviction that when thinking of mathematics in this university, when thinking of the contents of our courses, examinations and equipment, we should bear in mind the practical utility of mathematics, as also the elegance and beauty of abstract pure mathematics. We look forward to a time when students from this university may go out to use their knowledge to the social well being of our people, be it by the applied mathematics of engineering and industry, or by mathematical statistics in matters of social and economic planning, or of astronomy and aerodynamics in the art of weather forecasting. We are hoping in the mathematics department to have eventually at Peradeniya a small mathematical and statistical laboratory and an Observatory so that the practical side of mathematics may be encouraged. And at the same time, we hope too that there may go out of this university some students, who though not directly using their mathematical knowledge, may find satisfaction in a training which implies thinking accurately and thinking hard, or may have seen a glimpse of the beauty and elegance of mathematical thought, and perhaps have obtained therefrom aesthetic satisfaction and an upliftment of the spirit.

I think I am wandering away from my central theme to which I now revert.

3. Intuition and Proof

When one looks back on the last fifty years of mathematical development, one notices a movement towards more rigour in mathematical thinking, with less reliance on intuition. In the new developments in Analysis due to Hardy and Littlewood, rigour at every step has been the aim. In proving a proposition the assumptions and hypotheses under which the proposition may be made are chosen so that these hypotheses are just and only just sufficient to ensure the truth of the conclusion. That is, with the least amount of assumptions one endeavours to prove the most general consequences. This outlook has led to many interesting theorems, and work along these lines still continues. This kind of Analysis is sometimes spoken of as the 'hard, sharp, narrow' kind as contrasted with the 'soft, vague, broad' kind of some American and German mathematicians.

Ramanujan, the great Indian mathematician with a very exceptional gift of mathematical intuition, had, while remaining isolated, invented certain theorems and identities. When he was discovered by the mathematical world and sent to Cambridge, Professor Hardy tried to get Ramanujan to *prove* some of his results. But Ramanujan's conception of proof was so different from Hardy's! If intuitively Ramanujan was convinced of the truth of the theorem, he was satisfied. But intuition while it helps to lead to these theorems is no alternative to mathematical proof. There are some well known unproved theorems in existence—for example Fermat's last theorem, which people have been trying to prove for the last three hundred years. The theorem asserts that no whole numbers or fractions x, y, z, exist such that $x^n + y^n = z^n$, where n is a whole number greater than 2. The theorem is found to be satisfied by every set of numbers that have been tried so far, but a proof has not been possible. Substantial prizes have been offered for a complete proof, but competent mathematicians say that there must be other easier ways of making money than this one.

It is said that Ramanujan would produce roughly one theorem or a series or an identity each day, and it kept Hardy busy for months trying to prove these rigorously. Out of some hundreds of Ramanujan's results only a few

were eventually found to be incorrect. When asked how he got at these wonderful results, Ramanujan would say that the goddess of Namakkal inspired him with the formulae in his dreams. What a tragedy for mathematics that Ramanujan died when yet in his thirties, and when his mathematical opportunities had hardly begun.

4. Foundations of Mathematics

Another noticeable development has been the revival of interest in the foundations of mathematics, both for the clearing up of apparent paradoxes and difficulties, and from the philosophical point of view. The problem of the consistency of mathematics had been thought of as satisfactorily settled by Weirstrass and Cauchy in the first half of the nineteenth century, but the problem again became acute at the end of the century due to the work of Cantor on infinite numbers and infinite sets. This work cast serious doubts on the foundations of mathematics, bringing to light apparent paradoxes and inconsistencies. These difficulties inspired Whitehead and Russell to a detailed study of mathematical logic which led to their monumental work—the *Principia Mathematica*. This publication showed up the great scope of symbolic reasoning, but there still remain many contraversial questions. A Dutch mathematician named Brouwer has put forward some revolutionary ideas of logic, in some respects departing from traditional Aristotilean logic. For example, he would abandon the 'law of excluded middle' that a thing either is or is not, as for example, a number is either a prime number or not a prime number. Brouwer and his school have built up a system of logic which has received support as well as criticism, and the final confirmation is yet to come.

5. Generalised Geometries and Algebras

Another exciting development has been the growth of generalised geometries and algebras. The technique of mathematics consists of setting up a body of appropriate axioms from which conclusions and consequences of interest may be deduced. One writer has said:

> 'Mathematicians are like lovers—grant a mathematician the least principle, and he will draw from it a consequence which you must also grant him, and from this consequence another'.

Deductive argumentation is their method. Let us take as an illustration the Euclidean geometry which we learnt at school. The name is misleading because it might suggest that this elaborate system of geometry was developed by one man named Euclid. It was the consolidated statement of work of some hundreds of years. This geometry is built on certain axioms. From the axioms various interesting theorems may be deduced. The axioms themselves cannot be proved, but have to be assumed. For over two thousand years the truth of these axioms had been taken to be more or less self-evident. But studies on axiomatic formulations have shown that it is possible to build up other consistent systems of geometry, based on axioms slightly different from those of Euclidean geometry. A Russian mathematician, Lobatchewsky, was the first to build up such a geometry, and some years later Reimann invented another. In these two geometries all the axioms of Euclidean geometry are assumed except the one called the parallel postulate which is replaced by an alternative axiom.

Mathematically all these systems of geometry have the same status, though some may be more suited than others for application to the physical world. A question like 'Which of these geometries is the true one?' or the still larger question such as 'What is truth?' is beyond the range of the pure mathematician. It is for this reason that the word 'truth' is gradually fading out of the vocabulary of the mathematician (and perhaps of the physicist too!). They speak only about validity, whether from a particular set of axioms a particular deduction is valid or what are the valid deductions that may be made. This is the type of question the mathematician is concerned to answer.

Just as we have these different kinds of geometries, there have also been invented new systems of algebras. The familiar algebra of ordinary numbers satisfies certain well known axioms or laws. For example, there are the commutative laws

$$a + b = b + a, ab = ba;$$

the distributive law

$$a (b + c) = ab + ac;$$

the associative laws

$$a + (b + c) = (a + b) + c, a (bc) = (ab) c.$$

There have been invented algebras for which some of these laws need not hold. As illustration, there are these entities called matrices which do not satisfy the commutative law $ab = ba$. In such an algebra if $ab = o$ we cannot infer that either $a = o$ or $b = o$. A great deal of work on these new algebras, and in the allied topic of theory of groups, is being carried on nowadays. The theory of groups is concerned with the study of symmetry. It is said that Egyptian architecture, such as that in the Pyramids, reveals an advanced knowledge of the properties of symmetry, and some writers have suggested that if the Egyptians did not know the theory of groups they were very near knowing it. In recent years group theory has been of great value in the study of atomic architecture—of the way in which the different parts of the atom are arranged, a topic to which I shall refer a little later on.

6. Relativity Theory

Most of these developments I have spoken about have been taken in hand for their mathematical interest, but sometimes some of these have found

unexpected applications in physics. That mathematics would prove a useful tool for discerning the laws of Nature was a firm conviction of the early Greeks. Plato is said to have remarked 'God ever geometrises', and in the outside of his academy were the words 'Let no man ignorant of geometry enter here'. Pythagoras who invented the mathematical notation of music saw a harmony behind the Universe which could best be expressed in mathematical language. In the fifteenth century Leonardo da Vinci wrote:

> 'There is no certainty where one can neither apply any of the mathematical sciences nor any of those based on the mathematical sciences'.

This was not the sentiment of a narrow specialist extolling his own speciality, but of one who has come down to us as a model of versatility. What was true of science in Leonardo's days is even more so now, in these days of relativistic mechanics and quantum physics.

The relativity theory of gravitation provides a good illustration of the way in which some branch of pure mathematics comes of use in physics. The non-Euclidean geometry developed by Reimann in the nineteenth century was made use of by Einstein for his theory of gravitation. Newton's theory of gravitation, so well known and so eminently successful, had one difficulty right from the beginning, and that is that it uses the notion of action at a distance, a force of a kind that is rather difficult to visualise. When two bodies are touching one another one can understand a force of reaction between them; or if a string is used to pull a body one can visualise the force of tension in the string. But force at a distance—between say the Earth and the Sun, millions of miles apart, is difficult to believe. Why one introduces the notion of force in such a case is because the path of the Earth as it moves along is observed to bend towards the Sun. Einstein took the step of abandoning this idea of force, and ascribed the bending to the geometric properties of the space round the Sun. When there is a large body like the Sun, its presence distorts the space round it, the space becomes curved, and therefore the Earth moving in the space round the Sun would move according to the geometry of this space. The curvature of this space gives rise to the curved path of the Earth.

To give his theory a mathematical basis, Einstein made use of Reimannian geometry. The Einstein theory and the Newtonian theory give in most cases the same results, to the available degree of approximation, but Einstein predicted a few experimental observations where the difference may be detected. Observations undertaken after the predictions have given the verdict in favour of the Einstein theory. But this Einstein theory of gravitation—the general theory of relativity as it is called—has still many complications in it, and is not a fully accepted scheme, and further work along these lines continues. The special theory relativity, needed for the description of fast moving systems in electromagnetic fields, is now a fully accepted scheme and is a corner stone in atomic theory. It was this theory that first suggested that a great deal of energy should be obtainable from atomic systems by transforming the energy of matter into other forms of energy.

7. Atomic Physics

Another illustration of the way in which Pure Mathematics unexpectedly comes of use in physics is provided by non-commutative algebra, about which I spoke earlier, and which is now invoked for the mathematical statement of the laws of atomic physics. In the study of matter, first there were studied the properties of substances as we encounter them in ordinary life. And the study proceeds by analysis of these substances when they are divided and sub-divided into smaller entities. Ordinary chemical division gives rise to the atoms as the ultimate units of matter, but physical processes have made possible further division. The chemical atoms have been shown to be really composite systems, composed of a few elementary particles. At first it was thought that there were only two such particles, and that out of them, all substances could be made. These were the electron and the proton. But in recent years there has been an alarming increase in their number. The existence of the neutron, the positron and positive and negative mesons has been well established by experiment, and the existence of a few other particles such as the neutrino has been postulated from various theoretical considerations.

It has been found that classical physics, the physics of Galileo, Newton and Faraday, is not able to describe the behaviour of these particles. For example, the electrons behave sometimes like particles and sometimes like waves, a paradoxical behaviour that could not be reconciled by the classical theory. The need for a new mechanics to describe these small atomic systems arises because such systems cannot be observed with as much detail as large systems. We cannot touch an electron to see its hardness, or look at it to see where it is. Our knowledge about such a particle is much more indirect. To give an illustration, suppose we want to find the position of an electron, we have to look through a microscope. Here a ray of light after reflection by the electron reaches our eye through the microscope. But the light ray on its encounter with the electron would have imparted some of its energy and momentum to the electron, so that the electron has been disturbed by the act of observation. Any further information that we can have will be only of the disturbed electron and not of the original state. The disturbance caused by the act of observation cannot be reduced beyond a certain limit, unlike in the case of large systems. If we want to know where the Moon is, we look at it, if necessary through a telescope. The fact that we are looking at it will not make any difference to the motion of the Moon. But not so with electrons and atoms !

Our approach to the laws of atomic physics has thus to be indirect. One has to take the well established laws of classical physics, and modify or reinterpret them in a way in which they could describe atomic events. The study of these laws is quantum mechanics, and its formulation makes use of non-commutative algebra.

A description of the atomic system is not always possible in terms of pictures or models as classical mechanics would allow, but we have to be content with a mathematical picture. If one is asked the question 'What is an electron ?', one may attempt to describe it by its physical properties and say it is a particle with such a charge, and such a rest mass, and it is a little spinning magnet and so on, but such description is only a partial one. The complete answer that is now available to the question 'What is an electron ?' is the uninspiring answer that it is what is described by the Dirac relativistic quantum mechanical wave equation.

8. Concluding Remarks

In this way one sees that mathematics is becoming increasingly indispensable for scientific description. That is why any attempt to teach science without mathematics is not going to be very successful. J. J. Thomson, a distinguished experimenter, gave that warning some years ago, to the effect that people who are trying to omit the harder mathematical portions in their science teaching will only succeed in teaching that part of science that wont give a headache even to a caterpillar.

I think I have spoken sufficiently at length to indicate, though so inadequately, that there is a great deal of work yet to be done in mathematics, that mathematics did not begin with Pythagoras or Archimedes and will not end with Brouwer or Dirac, that it has utility if that is wanted or beauty and elegance if that is preferred, and that students who have a liking for the subject will find in it, if they take it seriously enough, much scope for their energies and a subject well worth doing.

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