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ON A GEOMETRIC PROPERTY OF FINITE BLASCHKE PRODUCTS

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Let B be a finite Blaschke product of degree 3 with distinct zeros at 0, a and b, and let $|\lambda|=1$. Also, let x_1, x_2 and x_3 be the distinct solutions of $B(z)=\lambda$. In a recent paper in the American Mathematical Monthly, Daepp, Gorkin and Mortini proved that the sides of the triangle $x_1 x_2 x_3$ are tangent to the ellipse $|w-a|+|w-b|=|1-a\bar{b}|$, and presented an algebraic generalization of it using the following result as the main tool of the proof:

Let λ be a complex number with $|\lambda|=1$, and let B be a finite Blaschke product of degree n with n distinct zeros and with $B(0)=0$. If x_r is any point such that $B(x_r)=\lambda$, then there exists m_r with $0<m_r<1$ and a finite Blaschke product C of degree $n-1$ with $C(0)=0$ such that

$$\frac{B(z)/z}{B(z)-\lambda}=\frac{m_r}{z-x_r}+(1-m_r)\frac{C(z)/z}{C(z)-\lambda}. \quad \ldots\ldots\ldots\ldots (1)$$

In this study, we first show that the equation (1) follows easily from a known theorem, and give another proof of it using the Herglotz representation formula. Next, we apply this result to give a new proof of the following theorem established in the above paper:

If B is a finite Blaschke product of degree 2 with distinct zeros and with $B(0)=0$, then for any λ with $|\lambda|=1$, the line joining the distinct points x_1 and x_2 satisfying $B(x_1)=B(x_2)=\lambda$ passes through the non-zero zero of B.

Also, we extend the geometric properties established in Daepp *et. al.* paper by removing the conditions imposed on the zeros of the finite Blaschke product.