COCYCLES AND BILINEAR FORMS

D.G.T.K. Samarasiri and A.A.I. Perera.

Department of Mathematics, Faculty of Science, University of Peradeniya

Introduction

A *bilinear form* on a vector space V is a mapping $\alpha : V \times V \rightarrow F$,where F is a field, satisfying,

1. $\alpha(u+u',v)=\alpha(u,v)+\alpha(u',v)$,
2. $\alpha(u,v+v')=\alpha(u,v)+\alpha(u,v')$,
3. $\alpha(\lambda u,v)=\alpha(u,\lambda v)=\lambda\alpha(u,v)$.

Any bilinear form on F^n can be expressed as

$\alpha(x,y)=x^T Ay=\sum_{i,j=1}^{n} a_{ij}x_i y_j$, where A is an $n \times n$ matrix whose elements are in F. If G is a finite group and C is a finite abelian group, a 2-cocycle is a mapping

$\varphi(g,h)\varphi(gh,k)=\varphi(g,hk)\varphi(h,k)\forall g,h \in G$.

This implies

$\varphi(g,1)=\varphi(1,h)=\varphi(1,1) \quad \forall g,h \in G$

A cocycle φ is naturally displayed as a cocyclic matrix; that is, a square matrix whose rows and columns are indexed by the element of G under some fixed ordering and whose entry in position (g,h) is $\varphi(g,h)$. The matrix $M_\varphi=[\varphi(g,h)]_{g,h\in G}$ is called a G-cocyclic matrix over C. Some authors call this matrix a pure cocyclic matrix (Flannery, 1996).

In our work, we prove the following Lemma.

Lemma: Every bilinear form is an additive cocycle and not vice versa.

Methodology

A Cocyclic matrix over G is a Hadamard product of Inflation, Transgression and Coboundary matrices.(Horadom, 1993) Coboundary matrix of G over C can be obtained by normalizing the multiplication table of the group G and construction Inflation and Transgression matrices are given by K.J. Horadom and W. De. Launey (1993).

Results

It can be easily shown that every bilinear form is an additive cocycle.

Let $\alpha : G\times G \rightarrow C$ be a 2-cocycle, then

$\alpha(x,y)\alpha(xy,z)=\alpha(x,yz)\alpha(y,z)$.

In additive form

$$\alpha(x,y)+\alpha(x+y,z)=\alpha(x,y+z)+\alpha(y,z)$$

If α is bilinear

$\alpha(x,y+z)=\alpha(x,y)+\alpha(x,z)$and

$\alpha(x+y,z)=\alpha(x,z)+\alpha(y,z)\forall x,y,z \in G$.

$\therefore \alpha(x,y)+\alpha(x+y,z)=\alpha(x,y)+\alpha(x,z)+\alpha(y,z)=\alpha(x,y+z)+\alpha(y,z)\,\forall x,y,z \in G$.

So, α satisfies a cocycle equation and hence it is a 2-cocycle.

We prove that the converse of this Lemma is not true, in general, by giving the following counter example.

Consider the finite group Z_2^3 which is a Z_2-module. Therefore, it is a vector space over Z_2.

Define a mapping $\alpha : Z_2^3 \times Z_2^3 \to Z_2$ such that
$\alpha(x,y)\alpha(xy,z) = \alpha(y,z)\alpha(x,yz)$ for all x,y,z in Z_2^3.

If α is bilinear it should satisfy the following:
$\alpha(x, y+z) = \alpha(x,y) + \alpha(x,z)$; and
$\alpha(x+y, z) = \alpha(x,z) + \alpha(y,z)$
for all x, y, z in Z_2^3.

First we will compute the cocyclic matrix for Z_2^3.

Inflation matrix for Z_2^3 is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & A & 1 & A & 1 & A & 1 & A \\ 1 & 1 & B & B & 1 & 1 & B & B \\ 1 & A & B & AB & 1 & A & B & AB \\ 1 & 1 & 1 & 1 & C & C & C & C \\ 1 & A & 1 & A & C & AC & C & AC \\ 1 & 1 & B & B & C & C & BC & BC \\ 1 & A & B & AB & C & AC & BC & ABC \end{pmatrix}$$

where $A^2 = B^2 = C^2 = 1$.

Transgression matrix for Z_2^3 is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & K & 1 & K & 1 & K & 1 & K \\ 1 & K & 1 & K & 1 & K & 1 & K \\ 1 & L & M & LM & 1 & L & M & LM \\ 1 & L & M & LM & 1 & L & M & LM \\ 1 & KL & M & KLM & 1 & KL & M & KLM \\ 1 & KL & M & KLM & 1 & KL & M & KLM \end{pmatrix}$$

where $K^2 = L^2 = M^2 = 1$.

Coboundary matrix for Z_2^3 is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & P & P & Q & Q & R & R \\ 1 & P & 1 & P & S & QRS & S & QRS \\ 1 & P & P & 1 & PRS & PQS & PQS & PRS \\ 1 & Q & S & PRS & 1 & Q & S & PRS \\ 1 & Q & QRS & PQS & Q & 1 & PQS & QRS \\ 1 & R & S & PQS & S & PQS & 1 & R \\ 1 & R & QRS & PRS & PRS & QRS & R & 1 \end{pmatrix}$$

The cocyclic matrix for Z_2^3 is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & A & P & AP & Q & AQ & R & AR \\ 1 & KP & B & BKP & S & KQRS & BS & BKQRS \\ 1 & AKP & BP & ABK & PRS & AKPQS & BPQS & ABKPRS \\ 1 & LQ & MS & LMPRS & C & CLQ & CMS & CLMPRS \\ 1 & ALQ & MQRS & ALMPQS & CQ & ACL & CMPQS & ACLMQRS \\ 1 & KLR & BMS & BKLMPQS & CS & CKLPQS & BCM & BCKLMR \\ 1 & AKLR & BMQRS & ABKMPRS & CPRS & ACKLQRS & BCMR & ABCKLM \end{pmatrix}$$

Transforming to additive entries over $Z_2 = \{0,1\}$ one can obtain the following cocyclic matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & P & A+P & Q & A+Q & R & A+R \\ 0 & K+P & B & B+K+P & S & K+Q+R+S & B+S & B+K+Q+R+S \\ 0 & A+K+P & B+P & A+B+K & P+R+S & A+K+P+Q+S & B+P+Q+S & A+B+K+P+R+S \\ 0 & L+Q & M+S & L+M+P+R+S & C & C+L+Q & C+M+S & C+L+M+P+R+S \\ 0 & A+L+Q & M+Q+R+S & A+L+M+P+Q+S & C+Q & A+C+L & C+M+P+Q+S & A+C+L+M+Q+R+S \\ 0 & K+L+R & B+M+S & B+K+L+M+P+Q+S & C+S & C+K+L+P+Q+S & B+C+M & B+C+K+L+M+R \\ 0 & A+K+L+R & B+M+Q+R+S & A+B+K+L+M+P+R+S & C+P+R+S & A+C+K+L+Q+R+S & B+C+M+R & A+B+C+K+L+M \end{pmatrix}$$

Consider $Z_2^3 = \{000, 100, 010, 110, 001, 101, 011, 111\}$. Then, we can obtain its multiplication table under addition as follows:

$e = 000, a = 100, b = 010, a+b = 110, d = 001, a+d = 101, b+d = 011, a+b+d = 111.$

+	e	a	b	a+b	D	a+d	b+d	a+b+d
e	000	100	010	110	001	101	011	111
a	100	000	110	010	101	001	111	011
b	010	110	000	100	011	111	001	101
a+b	110	010	100	000	111	011	101	001
d	001	101	011	111	000	100	010	110
a+d	101	001	111	011	100	000	110	010
b+d	011	111	001	101	010	110	000	100
a+b+d	111	011	101	001	110	010	100	000

Now, $\alpha(a, b+d) = \alpha(100, 011) = R$

$\alpha(a, b) = \alpha(100, 010) = P$

$\alpha(a, d) = \alpha(100, 001) = Q$

In general, $R \neq P + Q$.

Therefore, α is not bilinear.

Therefore,

$\alpha(a, b+d) \neq \alpha(a, b) + \alpha(a, d)$.

Conclusion

We have proved that a bilinear form is an additive cocycle. Further, we proved that the converse is not true by giving a counter example.

References:

Flannery, D.L. (1996). Calculation of cocyclic matrices. Journal of Pure and Applied Algebra, 14: 181-190.

Horadom, K.J. and De Launey, W. (1993). Cocyclic development of designs. Journal of Algebraic Combinatorics, 2(3): 267-290.