CONSTRUCTION OF NEW GRAPHS USING STEINER TRIPLE SYSTEMS AND THEIR DECOMPOSITIONS

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Introduction

Steiner Triple systems are mostly used in Combinatorics which is a branch in mathematics. A Steiner triple system of order n, *STS(n),* is a pair (X,B) which consists of a set X of n points and set B of 3-element subsets of X (called triples or blocks), with the property that any two points of X lie in a unique triple.(Cameron, 1994).

Using Steiner triple systems one can construct new graphs, called Block Intersection Graph. The "Block Intersection Graph" of a Steiner triple system of order n denoted by *BIG(STS(n)),* is a graph with the triples in B being the vertices of the graph, and with an edge joining two of its vertices if and only if the corresponding triples contain a common point. Since a *STS(n)* has replication number , any *BIG(STS(n))* is clearly regular of degree . Moreover, each point in X will correspond to a unique clique in the *BIG(STS(n)),* and any two of these n cliques will intersect in precisely one vertex. In our work , the *BIG(STS(n))* for any *STS(n)* has been constructed and the BIG(STS(n)) has been decomposed into triangles (Horak and Rosa, 1984).

Methodology

Various partition of the triples in a *STS(n)* into small configuration are considered. In particular , one such is a "triangulation" of a *STS(n),* which is a partition of the triples into sets of three, any two of the three intersecting, but with no point common to all three triples. If the *STS(n)* has number of blocks, , not divisible by three, then either one or two triples are omitted from the partition, depending upon whether is *1* or *2 (mod 3)*. Thus three triples of the form $\{a,b,d,\},\{a,c,e\},\{b,c,f\}$ form a *"triangle"* in a possible triangulation, where the points a, b and c are in two of the three triples.

A triangulation of a *STS(n)* will correspond to a parallel class of triangles in the *BIG(STS(n)).* However, any triangle in a *BIG(STS(n))* does not necessarily correspond to such a *"triangle"* consisting of three triples as above. For instance, the three triples of the form $\{a,b,c\},\{a,d,e\},\{a,f,g\}$ will also correspond to a triangle in the *BIG(STS(n)),* although these three triples form a *"3-windmill"* and not a *"triangle"*.(Mullin *et al*., 1897).

When the BIG has an odd degree, a spanning subgraphof odd degree needs to be removed first. Depending upon the number of edges the BIG contains, this spanning subgraph is either a 1- factor, or it has one edge more than a 1-factor (is usually denoted by T for *tripole*), or else two edges more than a 1-factor. We refer to such a minimal set of unused edges in the *BIG* decomposition into triangles as the *leave*. The leaves other than a 1-factor are given in the following figures.

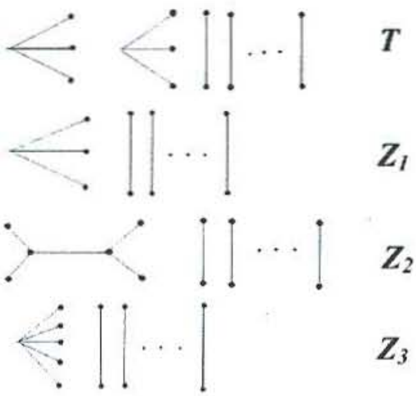

Figure 1. Spanning subgraphs of odd degree

There are three spanning subgraphs of odd degree having two edges more than 1-factor; we denote these by Z_1, Z_2 and Z_3..

When the BIG has even degree, since the *BIG(STS(n))* is regular of degree , $3(n-3)/2$ and since $n \equiv 1 \text{ or } 3$ (mod 6), the *BIG* has even degree precisely when $n \equiv 1 \text{ or } 7$ (mod 12).

Results

Considering the degree of the *BIG(STS(n))*, we can construct *BIG* for any *STS(n)*. For example consider the following constructions of block intersection graphs.

1. Since BIG has an even degree when $n=7$, *BIG(STS(7))* can be construct as follows. Let $\{0,1,2,3,4,5,6\}$ be the set of vertices of *STS(7)*. The triples of *STS(7)* are :

   $A=\{0,1,3\}$, $B=\{1,2,4\}$, $C=\{2,3,5\}$, $D=\{3,4,6\}$, $E=\{4,5,0\}$, $F=\{5,6,1\}$, $G=\{6,0,2\}$.

   So the vertices of *BIG(STS(7))* are *A,B,C,D,E,F,G* and *BIG(STS(7))* constructed by *A,B,C,D,E,F,G* is a new *STS(7)* and its geometric representation is given below:

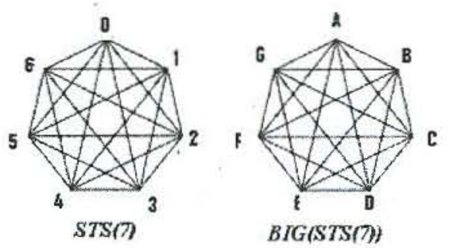

Figure 2. Construction of *BIG(STS(7))*

2. Consider the construction of *BIG(STS(13))* which has an odd degree. This shows that each of the three possible leaves can indeed be achieved. Consider the cyclic *STS(13)* formed by the sets $\{\{0,1,4\},\{0,2,7\}\}$.One possible triangulation of this *STS* is yielded by the following "triangles":

   $\{\{0,1,4\},\{1,2,5\},\{11,0,5\}\}$
   $\{\{1,3,8\},\{3,5,10\},\{4,5,8\}\}$
   $\{\{5,7,12\},\{6,7,10\},\{10,12,4\}\}$
   $\{\{7,8,11\},\{7,9,1\},\{9,11,3\}\}$
   $\{\{2,3,6\},\{3,4,7\},\{0,2,7\}\}$
   $\{\{2,4,9\},\{4,6,11\},\{5,6,9\}\}$
   $\{\{6,8,0\},\{8,9,12\},\{9,10,0\}\}$
   $\{\{12,1,6\},\{10,11,1\},\{11,12,2\}\}$

   The unused triples in the above triangulation are, and so this triangulation will yield a Z_1-leave.A second possible triangulation is yielded by the following "triangles";

   $\{\{0,1,4\},\{1,2,5\},\{11.0.5\}\}$

{{5,7,12},{6,7,10},{10,12,4}}
{{2,3,6},{3,4,7},{0,2,7}}
{{6,8,0},{8,9,12},{9,10,0}}
{{1,3,8},{3,5,10},{4,5,8}}
{{7,8,11},{7,9,1},{9,11,13}}
{{2,4,9},{4,6,11},{5,6,9}}
{{8,10,2},{10,11,1},{11,12,2}}

Since the unused two triples *{12, 0, 3}* and *{12,1,6}* have a point in common, we obtain a Z_2 leave.

Discussion

In the case when *BIG* has even degree, can be solved., while when the *BIG* has odd degree, removal of some spanning subgraphs of odd degree is necessary before the rest can be decomposed into triangles.

References

Cameron, P. J. (1994) Combinatorics topics techniques algorithms. Cambridge University Press, Cambridge.

Horak, P. and Rosa, A. (1984). Decomposing Steiner triple systems into small configurations. Springer , Berlin / Heidelberg.

Mullin, R.C., Popolove, A.L .and Zhu, L. (2008). Decomposition of Steiner triple systems into triangles. Journal of Combinatorial Mathematics, 48(3): 331-347.