Integer Solutions of Non - Linear Diophantine Equations Using Continued Fractions

A.G.K.N. Alupotha and A.A.I. Perera*

Department of Mathematics, Faculty of Science, University of Peradeniya

Introduction

The origin of continued fractions is traditionally placed at the time of the creation of Euclid's Algorithm. Euclid's Algorithm is used to find the greatest common devisor (gcd) of two numbers. However, by algebraically manipulating the algorithm, one can derive the simple continued fraction of a rational number p/q as opposed to the gcd of p and q. It is well-known that integer solutions of a linear Diophantine equation $ax+by=c,$ where $a,b,c \in Z,$ can be obtained by expressing a/b as a simple finite continued fraction. This idea can be generalized for the general linear Diophantine equation; $a_1x_1+a_2x_2+...+a_nx_n=b,$ where $a_1,a_2,...,a_n,b \in Z,$ can be solved by using continued fractions. Here, when n is large we need more conditions to solve the above Diophantine equation using continued fractions.

A Non-linear Diophantine equation of the form $x^2-dy^2=n$, where d and n are integers, is known as the more general Pell's equation, which can be solved for integer solutions by using continued fractions with additional two conditions. Further, this work has been generalized to a Non-linear Diophantine equation of the form $ax^2+bxy+cy^2+dx+ey+f=0$, where $a,b,c,d,e,f \in Z.$

Integer solutions to this equation can be obtained subject to three more conditions.

Method

Use Pell's equation; $x^2-dy^2=n$, where $d,n \in Z.$ Find integer solutions to Pell's equation when $d>0$ and $\sqrt{d} \neq m$ for all $m \in N$.

The following theorem can be used to find the integer solutions to $x^2-dy^2=1$.

Theorem: Assume that $d\,(>0)$ is not a square of an integer and that p_k/q_k is the k^{th} convergent of $\sqrt{d}$. Let s be the period of the simple continued fraction representation of $\sqrt{d}$.

(a) The positive integer solutions of $x^2-dy^2=1$ are $(x,y)=(p_{is-1},q_{is-1})$ for $i \geq 1$ when s is even and $(x,y)=(p_{2is-1},q_{2is-1})$ for $i \geq 1$ when s is odd.

(b) The positive integer solutions of $x^2-dy^2=-1$ are nonexistent when s is even and $(x,y)=(p_{(2i-1)s-1},q_{(2i-1)s-1})$ for $i \geq 1$ when s is odd, where the two sequences $\{p_n\}$ and $\{q_n\}$ are given by,

$p_0=a_0,\; p_1=a_0a_1+1,\; p_k=a_kp_{k-1}+p_{k-2};k\geq 2$ and
$q_0=1,\; q_1=a_1,\; q_k=a_kq_{k-1}+q_{k-1};k\geq 2.$

(Anderson and Bell, 1997)

This can be extended to find the integer solutions to $x^2-dy^2=n$ by writing

$$n=1.n=\left[(x^*)^2-d(y^*)^2\right](r^2-ds^2)$$
$$=(x^*r\pm dy^*s)^2-d(x^*s\pm y^*r)^2,$$

where (r,s) is a particular solution to $x^2-dy^2=n$ and (x^*,y^*) is an integer solution to $x^2-dy^2=1$. Thus $r^*=x^*r\pm dy^*s$ and $s^*=x^*s\pm y^*r$ is an integer solution to $x^2-dy^2=n$. Since we have infinitely many integer solutions for x^* and y^*, infinitely many integer solutions to our original Pell's equation can be obtained.

Now consider a Non-linear Diophantine equation of the form $ax^2+bxy+cy^2+dx+ey+f=0$, where $a,b,c,d,e,f \in Z.$ Integer solutions to this equation can be obtained with the following conditions:

(a) $a,c \neq 0$

(b) $b^2-4ac>0$

(c) $\sqrt{4(b^2-4ac)} \neq m$ for all $m \in N$.

Reduce this equation into a Pell's equation of the form $z^2-M\omega^2=N,$ and then obtain the integer solution for it. Hence, infinitely many integer solutions for x and y can be obtained.

Result

Consider $ax^2 + bxy + cy^2 + dx + ey + f = 0$.

Solution $x = \dfrac{-(by+d) \pm \sqrt{L}}{2a}$,

where $L = (by+d)^2 - 4a(cy^2 + ey + f)$.

Take $L = \omega^2 = (by+d)^2 - 4a(cy^2 + ey + f)$ for some integer ω. Then $Ay^2 + By + C = 0$, where $A = b^2 - 4ac, B = 2bd - 4ae$, $C = D - \omega^2$ and $D = d^2 - 4af$.

Take $z^2 = B^2 - 4AC = B^2 - 4A(D - \omega^2)$.

Then, $z^2 - M\omega^2 = N$, where $M = 4A$ and $N = B^2 - 4AD$. Solving this Pell's equation, integer solutions for z and ω can be obtained. Hence, we can find infinitely many integer solutions for x and y.

For an example, consider $x^2 + 3xy + y^2 + 5x + y + 1 = 0$.

Then, $a = 1, b = 3, c = 1, d = 5, e = 1$ and $f = 1$.

Since, $b^2 - 4ac = 5\ (>0)$ and $\sqrt{4(b^2 - 4ac)} = \sqrt{20} \notin$ N, the above method can be applied, where $A = 5, B = 26, C = 21 - \omega^2$, $D = 21, M = 20$ and $N = 256$. Next we shall find integer solutions to $z^2 - 20\omega^2 = 256$. Then, $(z_1, \omega_1) = (144, 32)$ will be an integer solution to $z^2 - 20\omega^2 = 256$.

Hence, $y = \dfrac{-B \pm z_1}{2A} = \dfrac{-26 \pm 144}{10} = -17$ or $\dfrac{118}{10}$.

Therefore, $y = y_1 = -17$ and hence, $x = \dfrac{-by_1 - d \pm \omega_1}{2a} = 7$ and 39.

Then, $(39, -17)$ and $(7, -17)$ are integer solutions to $x^2 + 3xy + y^2 + 5x + y + 1 = 0$.

Thus, we can find infinitely many integer solutions for given Non-linear Diophantine equation.

Discussion

If $\sqrt{d} = m$ for some $m \in$ N in $x^2 - dy^2 = n$, then $s = 0$. Since p_{-1} and q_{-1} are not defined $\sqrt{d}$ cannot be a natural number.

Integer solutions of the Diophantine equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$, can be obtained using continued fraction under some conditions. Further, we have proved that these conditions are necessary to obtain the solutions.

References

Anderson, J.A. and Bell, J.M. (1997) *Number theory with applications,* Prentice-Hall, New Jersy.