Computational Efficiency of Weakly Orthogonal Spherical Harmonics in Cubed Sphere

M.A.A.M. Faham[1], H.M. Nasir[2*] and T. Kaneshalingam[1]

[1]*Postgraduate Institute of Science,* [2]*Department of Mathematics, Faculty of Science, University of Peradeniya*[2]

Introduction

Numerical computations in solving problems defined on the sphere suffer from many difficulties near the poles, known as the 'pole problems', when using spherical polar coordinate system for the spherical surface. For example, in the computation of global weather prediction models, concentrated grid points near the poles increase the amount of computations in the pole region where quantities of interest are of less important than the other parts of the globe.

Avoiding pole problems has attracted some researchers in the recent past (Nasir and Faham, 2006). Among the recent developments in this direction, one of the present authors has constructed weakly orthogonal and orthogonal spherical harmonics in a non-polar spherical co-ordinate system based on the 'cubed sphere' defined from the surface of a unit cube (Nasir, 2007).

Computational efficiency is also one of the main targets in spherical transforms, which has been considered for spherical harmonics in *spherical polar coordinates by some* researchers (MohlenKamp, 1999). In this work, more properties and results for the orthogonal and weakly orthogonal spherical harmonics on cubed sphere are established. Computational efficiency of using weakly orthogonal spherical harmonics verses its orthogonal counterpart is analyzed. The Fourier series computations using the weakly orthogonal spherical harmonics are considered.

Weakly orthogonal and orthogonal spherical harmonics

A set of solutions for the eigen-value problem $\Delta_s u(t_1,t_2) = -l(l+1)u(t_1,t_2)$, where

$$\Delta_s = s_t\left((1+t_1^2)\frac{\partial^2}{\partial t_1^2} + 2t_1t_2\frac{\partial^2}{\partial t_1 \partial t_2} + (1+t_2^2)\frac{\partial^2}{\partial t_2^2} + 2t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2}\right)$$

and $s_t = 1+t_1^2+t_2^2$, is given by $y_l^{(m,n)} = \frac{p_l^{(m,n)}}{s_t^{1/2}}$,

$m+n=l-1,l$, where $p_l^{(m,n)}$ are polynomials in t_1 and t_2 satisfying the differential equation

$$(1+t_1^2)\frac{\partial^2 p}{\partial t_1^2} + 2t_1t_2\frac{\partial^2 p}{\partial t_1 \partial t_2} + (1+t_2^2)\frac{\partial^2 p}{\partial t_2^2} - (l-1)\left(2t_1\frac{\partial p}{\partial t_1} + 2t_2\frac{\partial p}{\partial t_2} - lp\right) = 0$$

and are explicitly given by the non-zero real or imaginary parts of

$$\sum_{p=m_1}^{m}{}_2\sum_{q=n_1}^{n}{}_2\binom{(m-p+n-q)/2}{(m-p)/2}\binom{l}{p,q}t_1^p t_2^q\, i^{(l-p-q)},\quad i=\sqrt{-1}$$

with $m_1 = m \bmod 2, n_1 = n \bmod 2$ and the subscript 2 in the summation indicates that the index variables increases with step 2.

A set of continuous spherical harmonics can be constructed from the eigen-functions for one face forming a six-tuple of functions

$$Y(t_1,t_2) = \left(y(t_1,t_2), y\left(\tfrac{-1}{t_1},\tfrac{-t_2}{t_1}\right), y\left(\tfrac{-t_1}{t_2},\tfrac{-1}{t_2}\right), y(t_1,-t_2), y\left(\tfrac{-1}{t_1},\tfrac{t_2}{t_1}\right), y\left(\tfrac{t_1}{t_2},\tfrac{-1}{t_2}\right) \right) \tag{1}$$

where, each component is an eigen-function. The spherical harmonics $y_l^{(m,n)}$ are weakly orthogonal in the sense that they are orthogonal for distinct l, but are not orthogonal among the $2l+1$ functions for a mode l.

Following the method of Kwon *et al.*, (2001) with some modifications, a set of completely orthogonal spherical harmonics can be constructed from the eigen solutions in the form

$$z_{l,r}^{(k)}(t_1,t_2) = \frac{Q_{l,r}^{(k)}}{s_t^{1/2}},\quad k=0,1$$

with $Q_{l,r}^{(k)} = p_r^{(k)}(t_1)\,\rho^{l-r}(t_1)\,q_{l-r}(l;\lambda)$, where $q_{l-r}(l;\lambda)$ and $p_r^{(k)}$ satisfy the differential equations

$$(1+\lambda^2)\frac{d^2q}{d\lambda^2}-(2l-1)\lambda\frac{dq}{d\lambda}+(l-r)(l+r)q=0 \text{ and}$$

$$(1+t_1^2)\frac{d^2p}{dt_1^2}-2(r-1)t_1\frac{dp}{dt_1}+r(r-1)p=0$$

respectively and $\rho(t_1)=(1+t_1^2)^{1/2}$ and $\lambda = t_2/\rho(t_1)$. The polynomials $p_r^{(k)}$ are given by the real and imaginary parts of $(t_1+\mathrm{i})^r$ and the polynomials $q_{l-r}(l;\lambda)$ are given by

$$q_{l-r}(l;\lambda)=\sum_{n=r_1}^{l-r}{}_2a_{l,n}^{(l-r)}\lambda^n,$$

where the coefficients $a_{l,n}^{(l-r)}$ satisfy

$$a_{l,n}^{(l-r)} = -[(n+1)(n+2)/(l-r-n)(l+r-n)]a_{l,n+2}^{(l-r)},$$

with $n=l-r-2, l-r-4, \cdots, r_1$ and $r_l=(l-r)$ mod 2.

We normalize the polynomials such that the leading coefficient is $a_{l,l-r}^{(l-r)}=\binom{l}{l-r}$.

The orthogonal spherical harmonics are then given by the six-tuple formed by $Z_{l,r}^{(k)}(t_1,t_2)$ as in (1).

We define the column vector P_l of size $2l+1$ of the polynomials $P_l^{(m,n)}$ for mode l and the column vector of polynomials Q_l corresponding to the orthogonal spherical harmonics as follows:

$$\begin{aligned}\mathrm{P}_l=[&P_l^{(0,l)},P_l^{(2,l-2)},\cdots,P_l^{(l-l_1,l_1)};\\&P_l^{(1,l-1)},P_l^{(3,l-3)},\cdots,P_l^{(l-1+l_1,1-l_1)};\\&P_l^{(0,l-1)},P_l^{(2,l-3)},\cdots,P_l^{(l-2+l_1,1-l_1)};\\&P_l^{(1,l-2)},P_l^{(3,l-4)},\cdots,P_l^{(l-1-l_1,l_1)}]^T\end{aligned}$$

and

$$\begin{aligned}\mathrm{Q}_l=[&Q_{l,0}^{(0)},Q_{l,2}^{(0)},\cdots,Q_{l,l-l_1}^{(0)};\\&Q_{l,1}^{(0)},Q_{l,3}^{(0)},\cdots,Q_{l,l-1+l_1}^{(0)};\\&Q_{l,1}^{(1)},Q_{l,3}^{(1)},\cdots,Q_{l,l-1+l_1}^{(1)};\\&Q_{l,2}^{(1)},Q_{l,4}^{(1)},\cdots,Q_{l,l-l_1}^{(1)}]^T\end{aligned}$$

where $l_1 = l \bmod 2$.

Each column vector of polynomials has four sets of components such that the first two sets correspond to the polynomials of degree l, while the second two sets correspond to the polynomials of degree l-1. The order of the column vectors is chosen so that their relations have a simple structure.

The column vectors of the weakly orthogonal and orthogonal spherical harmonics are then given respectively by

$$\mathrm{Y}_l=\frac{\mathrm{P}_l}{S_l^{1/2}},\quad \mathrm{Z}_l=\frac{\mathrm{Q}_l}{S_l^{1/2}}.$$

Linear relations:

We find that the relationship between the two set of polynomials is given by

$$Q_{l,r}^{(0)}=\frac{1}{2^{l-r}}\sum_{n=r_1}^{l-r}(-1)^{\frac{l-r-n}{2}}2^n\binom{l-n}{(l-r-n)/2}P_l^{(l-n,n)} \quad \text{and}$$

$$Q_{l,r}^{(1)}=\frac{1}{2^{l-r}}\sum_{n=r_1}^{l-r}(-1)^{\frac{l-r-n}{2}}2^n\frac{r}{l-n}\binom{l-n}{(l-r-n)/2}P_l^{(l-1-n,n)}$$

where $Q_{l,r}^{(k)}$, $k=0,1$ are orthogonal polynomials of degree l and l-1 respectively. In matrix form, we can write $\mathrm{Q}_l=\mathrm{T}_l\mathrm{P}_l$ and hence for the spherical harmonics

$$\mathrm{Z}_l=\mathrm{T}_l\mathrm{Y}_l \tag{2}$$

Linear relations among functions in the Six faces

Since each function in the six-tuple are eigen-functions of the Laplace-Beltrami eigen-value problem, each term is linearly related by functions defined on one face. To find these relations, we first derived the relation between $y_l^{(m,n)}(t_1,t_2)$ and $y_l^{(m,n)}(1/t_1,t_2/t_1)$,

$y_l^{(m,n)}\left(\frac{1}{t_1},\frac{t_2}{t_1}\right)$ where

$$=\sum_{q=n_1}^{n}{}_2\binom{(m-m_1+n-q)/2}{(m-m_1)/2}i^{(l-m_1-q)}\,y_l^{(l-m_1-q,q)}(t_1,t_2),$$

$m+n=l-1, l$. Relations for the functions in the six-tuple can then be obtained in view of the symmetric properties. The components of the six-tuple are then can be calculated.

Inner product

The norms of the orthogonal system of spherical harmonics are given by

$$\left\|Z_{l,r}^{(k)}\right\|^2=\frac{2^{2r+1}}{2l+1}\frac{(l!)^2}{(l-r)!(l+r)!}\begin{cases}\pi, & r>0\\ 2\pi, & r=0\end{cases} \tag{3}$$

Fourier series

The Fourier series of a spherical function $f(t_1;t_2)$ is written in terms of the weakly orthogonal spherical harmonics as

$$f(t_1,t_2)=\sum_{l=0}^{\infty}A_l\mathrm{Y}_l$$

where A_l is a row vector of size $2l+1$ of coefficients given by $A_l=\langle f,\mathrm{Y}_l\rangle\langle \mathrm{Y}_1,\mathrm{Y}_l\rangle^{-1}$, where $\langle \mathrm{Y}_1,\mathrm{Y}_l\rangle^{-1}=T_l^T\langle \mathrm{Z}_1,\mathrm{Z}_l\rangle^{-1}T_l$ by the use of (2).

We note that the vector of coefficients A_l is easily computable since the inner product matrix involving Z_l is diagonal and hence its inverse is obtained directly by the reciprocals of the inner products of $Z_{l,l-r}^{(k)}$ in (3).

Computational efficiency

Computations of the Fourier series representation of spherical functions involve truncation of the Fourier series to a finite series and computing it at discrete points. Computing the Fourier coefficients $\langle f,Y_l^{(m,n)}\rangle$ is also done by quadrature rules that require evaluation of the spherical harmonics at discrete points. Here, we see that computing weakly orthogonal spherical harmonics for a direct computation of discrete spherical harmonics requires less than 3 percent of total computational and storage cost.

Conclusions

We have established the linear relation between weakly orthogonal and orthogonal spherical harmonics. Hence, it is concluded that use of weakly orthogonal spherical harmonics is computationally cheaper than using its orthogonal counterpart. We also derived the linear relations between the eigen-functions in the six-tuple. Finally, norm of the spherical harmonics is found.

Acknowledgements

This research was partially supported by the Peradeniya University research grant (RG/2006/48/S).

References

Faham, M.A.A.M. and Nasir, H.M. (2006) Orthogonal spherical harmonics in cubed spherical Coordinates and applications to poisson equation in 2-sphere, *Proceedings of the PURSE*, University of Peradeniya.

Nasir, H.M. (2007) Spherical harmonics transform in a non-polar coordinate system and application to Fourier series in 2-sphere, *Mathematical Methods in the Applied Sciences, John-Wiley,* 30(14), 1843-1854.

Kwon, K.H., Lee, J.K. and Littlejohn, L.L. (2001) Orthogonal polynomial eigenfunctions of second order partial differential equations, *Transactions of the Mathematical Society,* 353(9), 3629 3647.

MohlenKamp, M.J. (1999) A fast transform for spherical harmonics, *Journal of Fourier Analysis and Applications,* 5, 159-184.